Numerical Methods in Radiative Transfer

Introduction to DOM, FVM and MCM

Cyril Caliot

PROMES-CNRS Lab.

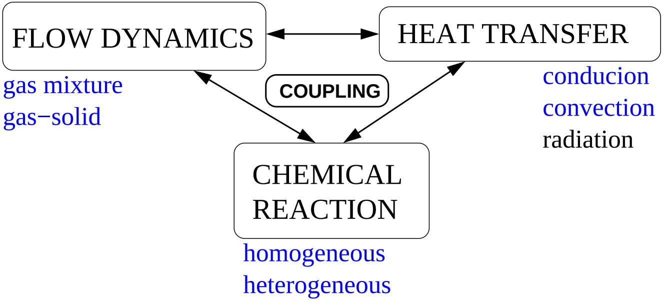
11 june 2010

Table of Contents

Table of Contents

Influence of Radiation

1. **In solar receptors-reactors, heat and mass transfer modes are coupled.**



Influence of Radiation

1. **In solar receptors-reactors, heat and mass transfer modes are coupled.**
2. At high temperature the heat transfer by **radiation becomes predominant**.

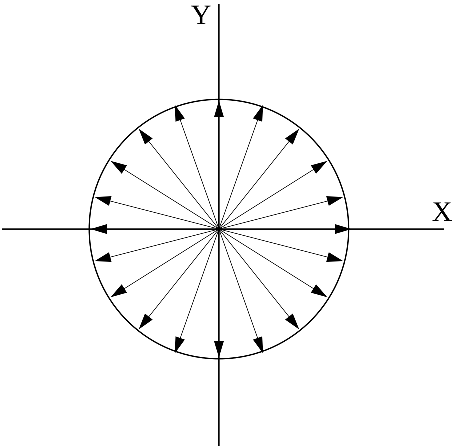
Influence of Radiation

1. **In solar receptors-reactors, heat and mass transfer modes are coupled.**
2. At high temperature the heat transfer by **radiation becomes predominant**.
3. To predict the influence of radiation one needs **a model for radiative heat transfer**
    - to compute the **radiative fluxes** on the walls
    - and the **radiative source terms** in a participative medium (emits, absorbs or scatters radiation)

Dependencies of Radiation

Radiative transfer models have to deal with

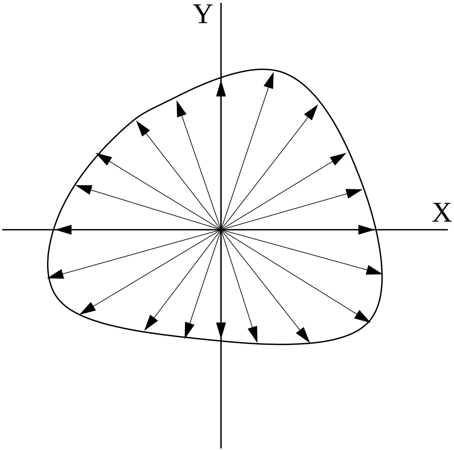
- the **angular** (or directional) dependency



Dependencies of Radiation

Radiative transfer models have to deal with

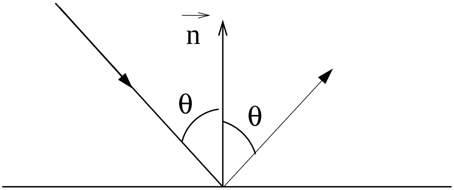
- the **angular** (or directional) dependency



Dependencies of Radiation

Radiative transfer models have to deal with

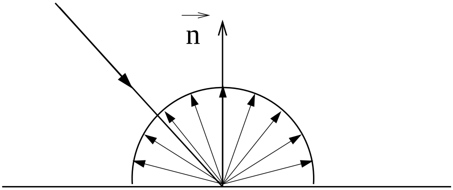
- the **angular** (or directional) dependency



Dependencies of Radiation

Radiative transfer models have to deal with

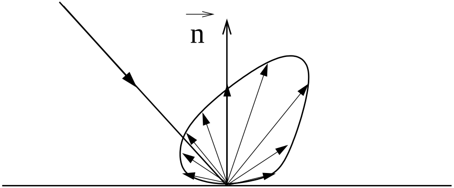
- the **angular** (or directional) dependency



Dependencies of Radiation

Radiative transfer models have to deal with

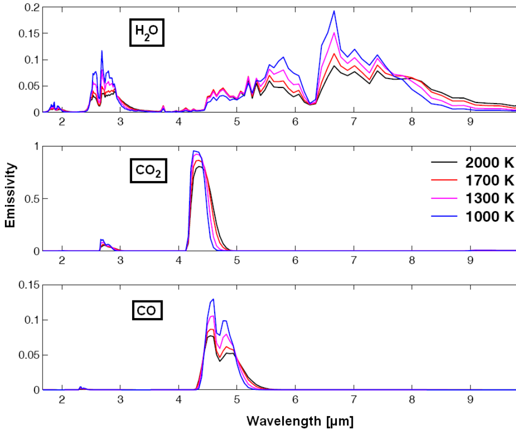
- the **angular** (or directional) dependency



Dependencies of Radiation

Radiative transfer models have to deal with

- the **angular** (or directional) dependency
- the **spectral** dependency



Radiative Transfer Equation

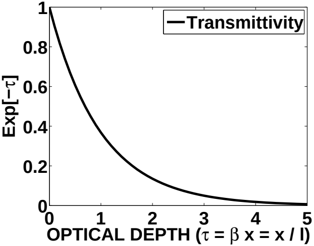
$$\vec{s}_i \cdot \overrightarrow{\nabla} I_\nu(\vec{r},\vec{s}_i) = -\beta_\nu I_\nu(\vec{r},\vec{s}_i) + \kappa_{a\nu} I_{b\nu}(T(\vec{r})) + \kappa_{s\nu} \int_{4\pi} I_\nu(\vec{r},\vec{s}_j) P_\nu(\vec{r},\vec{s}_j \rightarrow \vec{s}_i) d\Omega_j$$

$$\int_{4\pi} P_\nu(\vec{s}_i,\vec{s}) d\Omega_i \equiv 1$$

$$\kappa_{a\nu} = \frac{1}{l_{a\nu}} \quad ; \quad \kappa_{s\nu} = \frac{1}{l_{s\nu}}$$

$$\beta_\nu = \kappa_{s\nu} + \kappa_{a\nu} = \frac{1}{l_\nu}$$

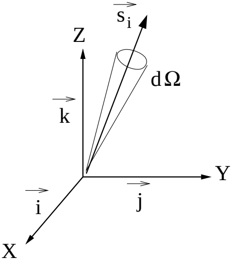
$$\omega_\nu = \frac{\kappa_{s\nu}}{\beta_\nu} = \frac{l_\nu}{l_{s\nu}} = 1 - \frac{\kappa_{a\nu}}{\beta_\nu}$$

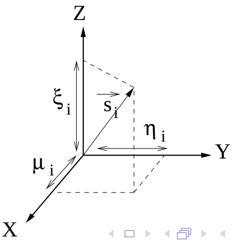


Radiative Transfer Equation

$$\vec{s}_i \cdot \overrightarrow{\nabla} I_\nu(\vec{r},\vec{s}_i) = \frac{dI_\nu(\vec{r},\vec{s}_i)}{ds} = \mu_i \frac{\partial I_\nu(\vec{r},\vec{s}_i)}{\partial x} + \eta_i \frac{\partial I_\nu(\vec{r},\vec{s}_i)}{\partial y} + \xi_i \frac{\partial I_\nu(\vec{r},\vec{s}_i)}{\partial z}$$

$$\begin{aligned}\vec{s}_i &= (\vec{s}_i \cdot \vec{i})\vec{i} + (\vec{s}_i \cdot \vec{j})\vec{j} + (\vec{s}_i \cdot \vec{k})\vec{k} \\ &= \mu_i \vec{i} + \eta_i \vec{j} + \xi_i \vec{k}\end{aligned}$$





Irradiation and Radiative Flux

The monochromatic incident radiation

$$G_\nu(\vec{r}) = \int_0^{4\pi} I_\nu(\vec{r},\vec{s})\, d\Omega$$

The monochromatic radiative flux vector

$$\vec{q}_{r\nu}(\vec{r},\vec{s}) = \int_0^{4\pi} I_\nu(\vec{r},\vec{s})\,\vec{s}\, d\Omega$$

The total radiative flux at a surface

$$q_{r,n}(\vec{r},\vec{s}) = \vec{q}_{r\nu}(\vec{r},\vec{s}) \cdot \vec{n} = \int_0^{\infty} d\nu \int_0^{4\pi} I_\nu(\vec{r},\vec{s}) \left(\vec{s}\cdot\vec{n}\right) d\Omega$$

Balance Equations

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

$$\frac{\partial (\rho \vec{v})}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v} \otimes \vec{v}) = -\vec{\nabla} p + \vec{\nabla} \cdot \vec{\vec{\tau}} + \rho \vec{f}$$

$$\frac{\partial (\rho e)}{\partial t} + \vec{\nabla} \cdot [\, (\rho e + p) \vec{v} \,] = \vec{\nabla} \cdot \left(\vec{\vec{\tau}} \cdot \vec{v} \right) + \rho \vec{f} \cdot \vec{v} - \underbrace{\vec{\nabla} \cdot \vec{q}}_{\text{Flux Divergence}} + R$$

$$\vec{\nabla} \cdot \vec{q} = \vec{\nabla} \cdot (\underbrace{\vec{q}_c}_{\text{Conductive}} + \underbrace{\vec{q}_r}_{\text{Radiative}})$$

$$\vec{\nabla} \cdot \vec{q}_c = \vec{\nabla} \cdot (-\lambda \vec{\nabla} T)$$

$$\vec{\nabla} \cdot \vec{q}_r = -\int_0^\infty \kappa_{a\nu} \left(\int_{4\pi} d\Omega \, I_\nu - 4\pi I_{b\nu} \right) d\nu$$

$$= -\int_0^\infty \kappa_{a\nu} (G_\nu - 4\pi I_{b\nu}) \, d\nu$$

Numerical Methods

Numerous methods are used in radiative transfer

- Spherical Harmonics Approximation : PN
- Discrete Ordinates Method : DOM
- Finite Volume Method : FVM
- Zonal Method : ZM
- Monte Carlo Method : MCM
- and others ...

Table of Contents

Angular Quadratures

The integrals over the directions are replaced by numerical quadratures :

$$\int_{4\pi} f \, d\Omega = \sum_{j=1}^{M} \Delta\Omega_j f(\vec{s}_j)$$

- $\vec{s}_j$ are the quadrature points (the discrete ordinates)
- $\Delta\Omega_j$ $(= w_j)$ are the quadrature weights (solid angle increments)

$$\begin{aligned} G_\nu(\vec{r}) &= \sum_{j=1}^{M} \Delta\Omega_j I_\nu(\vec{r},\vec{s}_j) \\ \vec{q}_r(\vec{r},\vec{s}) &= \int_0^\infty d\nu \sum_{j=1}^{M} \Delta\Omega_j I_\nu(\vec{r},\vec{s}_j)\vec{s} \end{aligned} \qquad (2)$$

Angular Quadratures

In the radiative transfer equation, the integral over the solid angle is approximated by division into increments.

$$\vec{s}_i \cdot \overrightarrow{\nabla} I_\nu(\vec{r},\vec{s}_i) = -\beta_\nu I_\nu(\vec{r},\vec{s}_i) + \kappa_{a\nu} I_{b\nu}(T(\vec{r})) + \kappa_{s\nu} \sum_{j=1}^{M} \Delta\Omega_j I_\nu(\vec{r},\vec{s}_j) P_\nu(\vec{r},\vec{s}_j \rightarrow \vec{s}_i)$$

The boundary condition (diffuse gray wall) becomes :

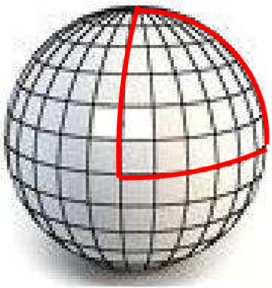
$$I_\nu(\vec{r}_w,\vec{s}_i) = \varepsilon_{\nu,w} I_{b\nu}(T(\vec{r}_w)) + \frac{\rho_{\nu,w}}{\pi} \sum_{\vec{n}_w \cdot \vec{s}_j < 0} \Delta\Omega_j I_\nu(\vec{r}_w,\vec{s}_j) |\vec{n}_w \cdot \vec{s}_j|$$

Symmetric Quadrature Sets

To **guarantee a solution invariance for any rotation of** 90^o around any coordinate axis the quadrature should have

1. **symmetric weights** and
2. **symmetric ordinates**.

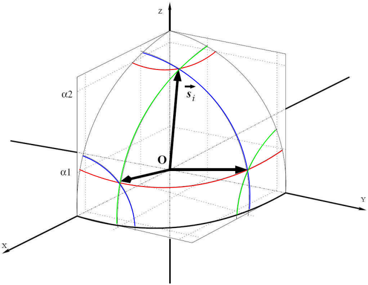
Description of the directions in one octant sufficies to describe the directions in all octants.



Angular Quadratures Sets

Many quadrature sets were developped :

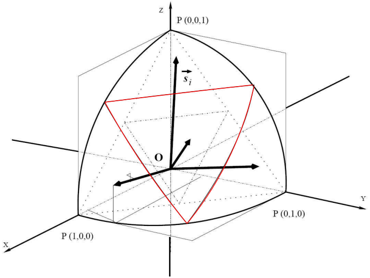
- the **level-symmetric S-N** quad. (S_4)



Angular Quadratures Sets

Many quadrature sets were developped :

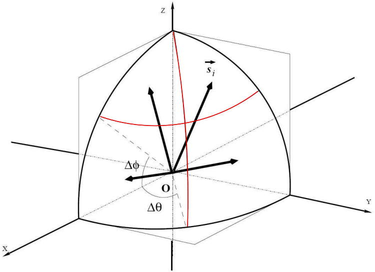
- the **level-symmetric S-N** quad. (S_4)
- the **T-N** quad. (T_2)



Angular Quadratures Sets

Many quadrature sets were developped :

- the **level-symmetric S-N** quad. (S_4)
- the **T-N** quad. (T_2)
- the **polar/azimuthal** angular quad. ($N_\theta = 2; N_\phi = 2$)

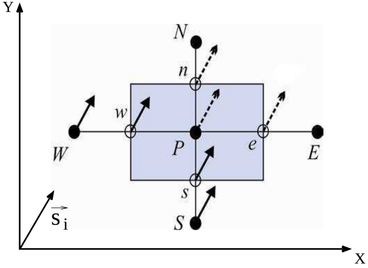


Discretization

To show how the DOM works we will consider

1. a **2D** domain with 2D volume elements and
2. a **cell-centered** discretization.

To solve the RTE, the **finite volume** method will be used.



Discretized RTE

The RTE is integrated over a control volume V_c, the LHS of RTE gives

$$\int_{V_c} \vec{s}_i \cdot \overrightarrow{\nabla} I_\nu(\vec{r},\vec{s}_i)\, dV = \int_{V_c} \overrightarrow{\nabla} \cdot \left(\vec{s}_i I_\nu(\vec{r},\vec{s}_i) \right) dV = \int_A I_\nu(\vec{r}_A,\vec{s}_i)\,(\vec{s}_i \cdot \vec{n})\, dA$$

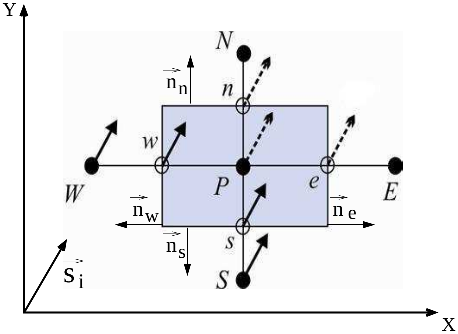
while the RHS gives

$$\int_{V_c} \left(-\beta_\nu I_\nu(\vec{r},\vec{s}_i) + S_\nu(\vec{r}_p,\vec{s}_i) \right) dV = V_c \left(-\beta_\nu I_\nu(\vec{r},\vec{s}_i) + S_\nu(\vec{r}_p,\vec{s}_i) \right)$$

with changes in the notation, the RTE becomes

$$\sum_{k=1}^{N_f} I_k^i\,(\vec{s}_i \cdot \vec{n}_k)\, A_k = V_c \left(-\beta I_p^i + S_p^i \right)$$

Closure Scheme



$$\sum_{k=1}^{N_f} I_k^i \left(\vec{s}_i \cdot \vec{n}_k\right) A_k = V_c \left(-\beta I_p^i + S_p^i \right)$$

$$\mu_i (A_e I_e^i - A_w I_w^i) + \eta_i (A_n I_n^i - A_s I_s^i) = V_p (S_p^i - \beta I_p^i)$$

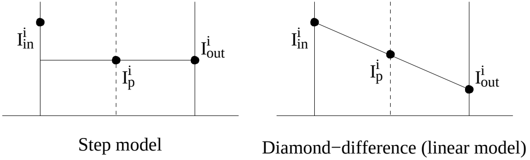
Need of a closure scheme such as the **step**, **diamond**, **CLAM** etc..

Spatial Discretization

The evolution of the intensity inside V_c is given by

$$\begin{aligned} I_p^i &= (1-\gamma)I_w^i + \gamma I_e^i \\ &= (1-\gamma)I_s^i + \gamma I_n^i \end{aligned}$$

For the step model $\gamma = 1$ and for the diamond scheme $\gamma = 0.5$.



$$\begin{aligned} I_e^i &= \frac{1}{\gamma}(I_p^i - (1-\gamma)I_w^i) \\ I_n^i &= \frac{1}{\gamma}(I_p^i - (1-\gamma)I_s^i) \end{aligned}$$

$$\mu_i(A_e I_e^i - A_w I_w^i) + \eta_i(A_n I_n^i - A_s I_s^i) = V_p(S_p^i - \beta I_p^i)$$

$$\begin{aligned} A_e I_e^i - A_w I_w^i &= \frac{1}{\gamma}\left(A_e I_p^i - I_w^i(A_e(1-\gamma) + \gamma A_w)\right) \\ A_n I_n^i - A_s I_s^i &= \frac{1}{\gamma}\left(A_n I_p^i - I_s^i(A_n(1-\gamma) + \gamma A_s)\right) \end{aligned}$$

$$I_p^i = \frac{\mu_i A_{ew} I_w^i + \eta_i A_{ns} I_s^i + \gamma S_p^i V_p}{\mu_i A_e + \eta_i A_n + \gamma\beta V_p}$$

$$\begin{aligned} A_{ew} &= (1-\gamma)A_e + \gamma A_w \\ A_{ns} &= (1-\gamma)A_n + \gamma A_s \end{aligned}$$

$$S_p^i = \kappa_a I_b(T_p) + \kappa_s \sum_{j=1}^{M} \Delta\Omega_j I_p^j P_p^{ji}$$

Convergence Method

Geometrical informations and T, P, X fields are given.

1. Choose a discrete ordinate $\vec{s}_i$
2. Compute I_p^i (in each cell)
3. Progess by a point-by-point iteration in the $\vec{s}_i$ direction

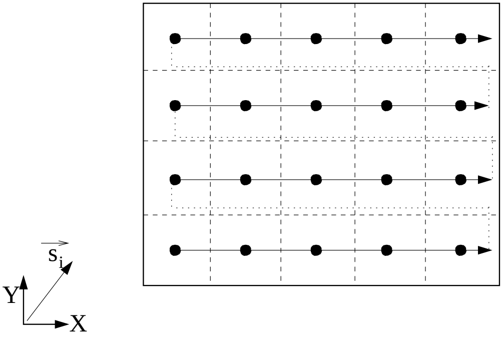


Table of Contents

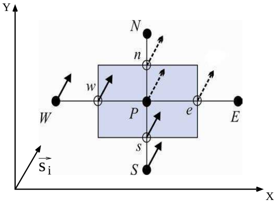
Domain Discretization

For a monochromatic RTE the space and angular variables are independent. In the FVM,

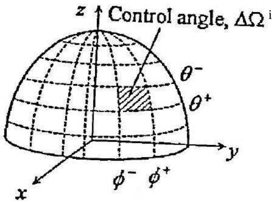
1. the **space** is subdivided into **control volumes** (CV).
2. the **solid angle** 4π is divided into M **control angles** (CA).

The directional intensity at node p, $I_{\nu p}(\vec{s}_i)$

- is obtained by **integrating the RTE** over a FV and a CA.
- is assumed **constant** over the FV and CA.



(a) **Control volume**



(b) **Control angle**

RTE Discretization

Integration of the LHS of the RTE (Green-Ostrogradski) gives

$$\int_{\Delta\Omega^i}\int_{V_c}\vec{s}_i\cdot\vec{\nabla} I_\nu(\vec{r},\vec{s}_i)\,dVd\Omega^i=\int_{\Delta\Omega^i}\int_{A} I_\nu(\vec{r}_A,\vec{s}_i)\,(\vec{s}_i\cdot\vec{n})\,dA\,d\Omega^i$$

while the RHS gives

$$\int_{\Delta\Omega^i}\int_{V_c}\Big(-\beta_\nu I_\nu(\vec{r},\vec{s}_i)+S_\nu(\vec{r},\vec{s}_i)\Big)\,dV\,d\Omega^i=\Big(-\beta_\nu I_\nu+S_\nu\Big)\,V_c\,\Delta\Omega^i$$

Discretized RTE

$$\sum_{k=1}^{N_{nb}} I^i_{nb}\,A_{nb}\int_{\Delta\Omega^i}(\vec{s}_i\cdot\vec{n}_{nb})d\Omega^i=\Big(-\beta I^i_p+S^i_p\Big)\,V_c\,\Delta\Omega^i$$

$$S^i_p=\kappa_a I_b(T_p)+\kappa_s\sum_{j=1}^{M} I^j_p\,\overline{P}^{ji}_p\,\Delta\Omega^j$$

RTE Discretization

$$\sum_{k=1}^{N_{nb}} I_{nb}^i A_{nb} \int_{\Delta\Omega^i} (\vec{s}_i \cdot \vec{n}_{nb}) d\Omega^i = \left(-\beta I_p^i + S_p^i \right) V_c \Delta\Omega^i$$

To solve the RTE a closure scheme such as the **upwind** (step), **diamond**, **CLAM** or **exponential** scheme (etc.) can be used. Then, the RTE can be written as

$$a_p^i I_p^i = \sum_{nb} a_{nb}^i I_{nb}^i + b^i$$

If the *upwind* scheme is used the coefficients are

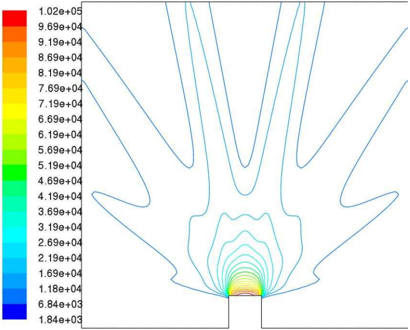
$$a_p^i = \sum_{nb} \max\left(A_{nb} D_{nb}^i, 0 \right) + \beta V_c \Delta\Omega^i \quad ; \quad D_{nb}^i = \int_{\Delta\Omega^i} (\vec{s}_i \cdot \vec{n}_{nb}) d\Omega^i$$

$$a_{nb}^i = \max\left(-A_{nb} D_{nb}^i, 0 \right) \quad ; \quad b^i = S_p^i V_c \Delta\Omega^i$$

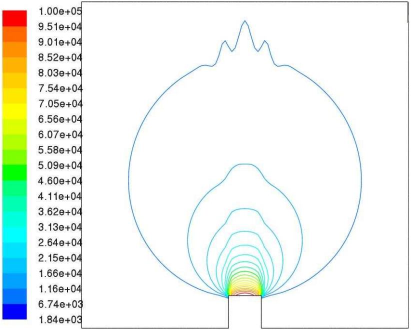
Table of Contents

Ray Effect

The **ray effect** is due to the **angular discretization**



(c) $N_\theta * N_\phi = 4 * 4$

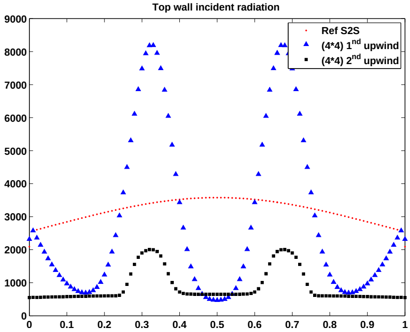


(d) $N_\theta * N_\phi = 20 * 20$

The remedy is to increase the number of discrete ordinates.

False Scattering

The **false scattering** is due to the **spatial differencing scheme**.



The remedy is to use a **better** spatial differencing scheme or a **finer mesh** (increase CPU time).

Table of Contents

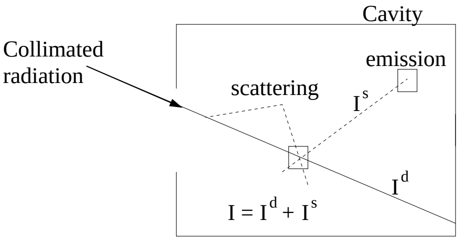
Intensity Splitting

A MDOM improves the treatment of the isolated boundary sources.

- **Aim :** Avoid the ray effect
- **Method :** Handle the direct boundary radiation separately from the medium radiation

The intensity is expressed as the sum of a direct and diffuse intensity

$$I(\vec{r},\vec{s}_i) = I_\nu^d(\vec{r},\vec{s}_i) + I_\nu^s(\vec{r},\vec{s}_i)$$



Direct Intensity

I_ν^d comes directly from a collimated or diffuse (wall) source.

$$\vec{s}_i \cdot \overrightarrow{\nabla} I_\nu^d(\vec{r},\vec{s}_i) = -\beta_\nu I_\nu^d(\vec{r},\vec{s}_i)$$

Directional contribution from a boundary

$$I_\nu^d(\vec{r},\vec{s}_i) = I_\nu^d(\vec{r}_w,\vec{s}_i)\exp\left[-\beta_\nu|\vec{r}-\vec{r}_w|\right]$$

Diffuse Intensity

I_ν^s is the remaining intensity which is emitted and scattered by the medium.

$$\vec{s}_i \cdot \overrightarrow{\nabla} I_\nu^s(\vec{r},\vec{s}_i) = -\beta_\nu I_\nu^s(\vec{r},\vec{s}_i) + S_\nu^s(\vec{r},\vec{s}_i)$$

with the source term defined by

$$\begin{aligned} S_\nu^s(\vec{r},\vec{s}_i) &= \kappa_{a\nu} I_{b\nu} + \kappa_{s\nu} \int_{4\pi} \left(I_\nu^d(\vec{r},\vec{s}_j) + I_\nu^s(\vec{r},\vec{s}_j) \right) P_\nu(\vec{r},\vec{s}_j \to \vec{s}_i)\, d\Omega_j \\ &= \kappa_{a\nu} I_{b\nu} + \kappa_{s\nu} \int_{4\pi} I_\nu^s(\vec{r},\vec{s}_j) P_\nu(\vec{r},\vec{s}_j \to \vec{s}_i)\, d\Omega_j \\ &\quad + \kappa_{s\nu} \int_{A_w} I_\nu^d(\vec{r}_w,\vec{s}_i) \exp[-\beta_\nu |\vec{r}-\vec{r}_w|] P_\nu(\vec{r},\vec{s}_j \to \vec{s}_i) \\ &\quad \frac{\vec{n}_w \cdot \vec{s}_i}{||\vec{r}-\vec{r}_w||^2}\, dA_w \end{aligned}$$

Treatment for Scattering

A MDOM improves the convergence when phase function with strong forward peak are considered.

- **Aim :** Reduce the number of iteration due to scattering
- **Method :** Remove the forward peak and treat it as transmission

Small changes : only the definitions of the extinction coefficient and the source term

$$S^i_{p,m} = \kappa_a I_b(T_p) + \kappa_s \sum_{\substack{j=1 \\ j\neq i}}^{M} \Delta\Omega_j I^j_p P^{ji}_p \quad (3)$$

$$\beta_m = \beta - \kappa_s \Delta\Omega_i P^{ii}_p \quad (4)$$

Table of Contents

Monte Carlo Estimator

The Monte Carlo method is used to estimate integrals such as :

$$S = \int_a^b f(x)\, p_X(x)\, dx = E(f(X))$$

Principle : produce a series of **independant random variables** $(x_1, x_2, \ldots, x_N)$, with $N \longrightarrow \infty$, from p_X on $[a, b]$ and compute the Monte Carlo estimate s_N $(E(s_N) = S)$:

$$s_N = \frac{1}{N} \sum_{i=1}^{N} f(x_i)$$

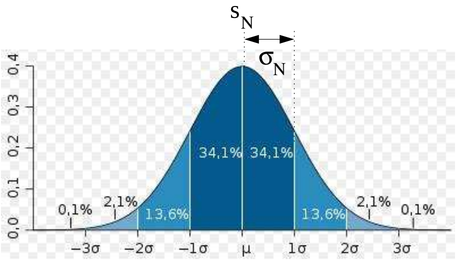
- p_X is a **probability density function** : uniform, $p_X(x) = \dfrac{1}{b-a}$
- s_N **Monte Carlo estimate**

Monte Carlo Error Estimate

Applying the Central Limit Theorem, the **standard error** of the mean s_N can be estimated by

$$\sigma_N = \frac{1}{\sqrt{N}} \sqrt{\frac{1}{N}\sum_{i=1}^{N} f(x_i)^2 - \left(\frac{1}{N}\sum_{i=1}^{N} f(x_i)\right)^2}$$

Error bars (confidence interval) : $P(s_N - \sigma_N\gamma \leq S \leq s_N + \sigma_N\gamma)$



$$P = 99\%, \gamma \approx 2.56$$

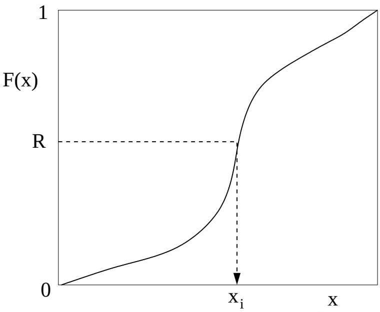
$$P = 95\%, \gamma \approx 1.96$$

Random Sampling

$\Re$ is uniformly distributed in $[0,1]$ then $x_i = b$ is randomly sampled with

$$F(X \leq b) = \int_{-\infty}^{b} p_X(x)\, dx = \Re$$

Probability density function :$p_X(x)$
$\forall x \in \mathbb{R}$, $p_X(x) > 0$ and $\int_{-\infty}^{+\infty} p_X(x)\, dx = 1$
Cumulative distribution function $F(x)$
(positive, monotone, non-decreasing and $\in [0,1]$)
$F(X \leq b) = \int_{-\infty}^{b} p_X(x)\, dx$



Next...

Questions and Practical Work on the Monte Carlo method